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A spacetime cloak, or a history editor

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Abstract

We introduce a new type of electromagnetic cloak, the spacetime cloak (STC), which conceals *events* rather than objects. Non-emitting events occurring during a restricted period are never suspected by a distant observer. The cloak works by locally manipulating the speed of light of an initially uniform light distribution, whilst the light rays themselves always follow straight paths. Any ‘perfect’ spacetime cloak would necessarily rely upon the technology of electromagnetic metamaterials, which has already been shown to be capable of deforming light in ways hitherto unforeseen—to produce, for example, an electromagnetic object cloak. Nevertheless, we show how it is possible to use intensity-dependent refractive indices to construct an approximate STC, an implementation that would enable the distinct signature of successful event cloaking to be observed. Potential demonstrations include systems that apparently violate quantum statistics, ‘interrupt-without-interrupt’ computation on convergent data channels and the illusion of a Star Trek transporter.

Keywords: electromagnetic cloaking, metamaterials, electromagnetic modulation, spacetime, observers and events

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Would it not be amazing if, in an age of ubiquitous surveillance, a method could be devised to somehow remove a piece of history that would otherwise be recorded by a surveying camera? A safe-cracker would be able, for a brief time, to enter a scene, open the safe, remove its contents, close the door and exit the scene, whilst the record of a surveillance camera apparently showed that the safe door was closed all the time. Although this sounds like science fiction, the lesson from metamaterials research in the last decade has taught us that, within certain restrictions, such speculations are not fantasy. We here show how the magic of editing history can be achieved by introducing the concept of the spacetime cloak (STC). The proposal opens up a new paradigm for electromagnetic cloaking and has major implications for research in metamaterials, slow light and phase modulation. Metamaterials already bend light in unconventional ways, producing aberration-free lenses [1] or

spatial ‘object’ cloaks, which coax electromagnetic radiation around a spatial void invisible to outside observers [2–4]. These latter developments are extensions of our historical and pre-historical abilities—that light can bend whilst traversing transparent materials has been known at least since humans went fishing with spears; even the controlled bending of light in prisms and lenses has been known for centuries. In contrast, the STC utilizes time-dependent nanometre-engineered electromagnetic metamaterials to create a temporal void within which events can be hidden. Remarkably, our spacetime ‘event’ cloak leaves light rays undeviated from their expected linear spatial trajectories, and instead works by dividing illuminating light into a leading part, which is speeded up and passes before a collection of events occur, and a trailing part, which is slowed down and passes after they have occurred. This process is thus utterly distinct from a spatial ‘object’ cloak, which instead achieves concealment by bending light around an object. The STC opens a temporary corridor through which energy, information and matter can be

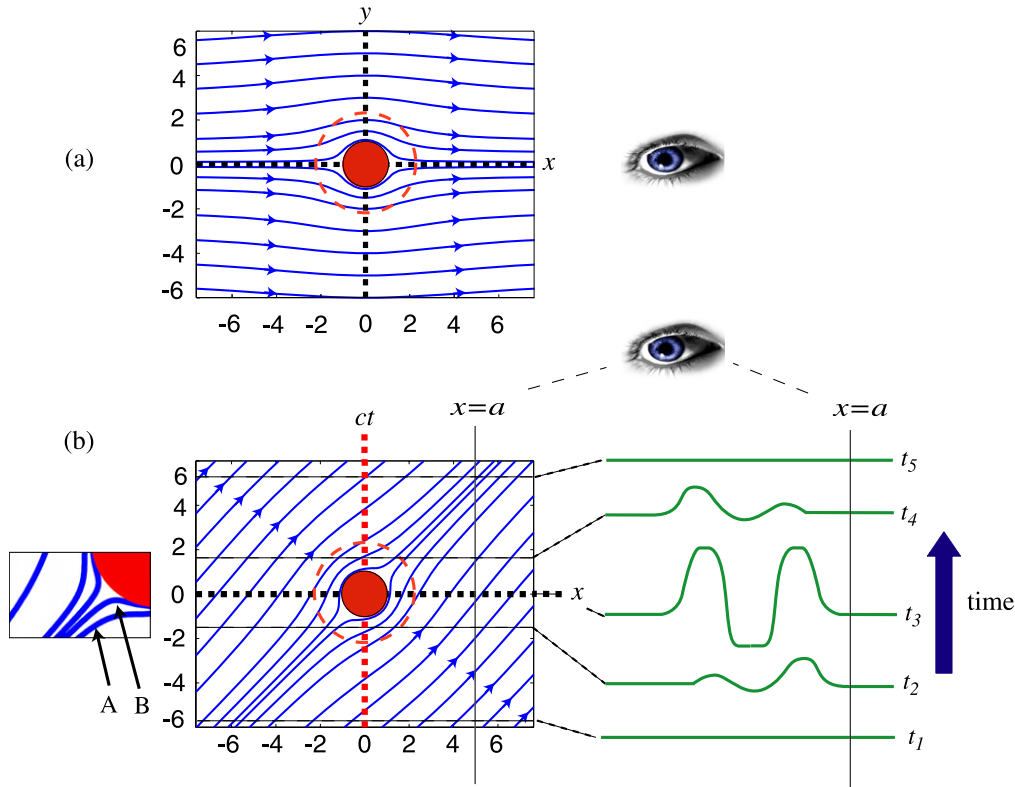


Figure 1. (a) Conventional spatial cloak based on a planar transformation $(x, y) \rightarrow (x', y')$. An observer located to the far right on the x axis does not see the object. (b) Spacetime cloak. An analogous coordinate transformation is used, but now in (x, t) rather than (x, y) . The cloak now conceals events near the spacetime origin. The (schematic) intensity distribution for various times is shown on the right, indicating the formation and subsequent evaporation of the intensity null that is fully developed at $t = 0$. The observer to the right never suspects the occurrence of any non-radiating events near the spacetime origin and sees a uniform intensity for all time.

manipulated or transported undetected. Once the concealed passage has been used, the STC closes by slowing the leading part of the light, whilst speeding up the trailing part, leaving no trace of the cloak, or the concealed events, on the field profile. To a distant observer, any non-emitting object whose presence persisted throughout the lifetime of the STC will have had a finite interval excised from its history. Any object which emits light during STC operation will have that part of its history temporally compressed so that, if not absorbed, it will appear to the observer as if occurring in a single instant. We here outline the theory and design of the spacetime cloak and describe potential applications: hiding events from some observers but not others, using the ‘time gap’ to generate apparent violations of quantum statistics, allowing the covert interruption of non-interruptible computation and/or signal processing operations, and creating the illusion of a Star Trek transporter.

2. The spacetime cloak concept

Our method for redacting history is based on applying the principles of transformation optics to produce an electromagnetic cloak working in both space and time, rather than in just space. Transformation optics relies on mimicking a coordinate transformation applied to Maxwell’s vacuum equations by means of an inhomogeneous dielectric/magnetic medium. A coordinate transformation $\mathbf{r} \rightarrow \mathbf{r}'$, taking straight

lines to curves, transforms the electromagnetic field so that in the new system

$$\nabla' \times \mathbf{E}' = -\frac{\partial \mathbf{B}'}{\partial t}, \quad \nabla' \cdot \mathbf{D}' = 0, \quad (1)$$

$$\nabla' \times \mathbf{H}' = \frac{\partial \mathbf{D}'}{\partial t}, \quad \nabla' \cdot \mathbf{B}' = 0. \quad (2)$$

Since the form of Maxwell’s equations is preserved, the vacuum relations $\mathbf{D} = \epsilon_0 \mathbf{E}, \mathbf{B} = \mu_0 \mathbf{H}$ are replaced in the new coordinate system by appropriate inhomogeneous, anisotropic relations $\mathbf{D}' = \epsilon'(\mathbf{r}') \cdot \mathbf{E}', \mathbf{B}' = \mu'(\mathbf{r}') \cdot \mathbf{H}'$. If instead of transforming coordinates, the original coordinate space (\mathbf{r}) is filled with a medium $\epsilon(\mathbf{r}), \mu(\mathbf{r})$, having the same inhomogeneous, anisotropic (and, strictly speaking, instantaneously responding) properties prescribed by $\epsilon'(\mathbf{r}'), \mu'(\mathbf{r}')$, then the linear light paths of vacuum become curved, just as straight lines become curved under $\mathbf{r} \rightarrow \mathbf{r}'$. Figure 1(a) shows how a spatial electromagnetic cloak [2] is produced via a time-invariant planar transformation $(x, y) \rightarrow (x', y')$ that diverts light rays so as to create a void near the origin. The polar transformation $r' = (1 - a/b)r + a, \theta' = \theta, z' = z$ distorts linear rays travelling in vacuum from left to right to those shown. Equivalently, rays in an anisotropic dielectric with polar parameters $(\epsilon, \mu)_r = (1 - a/r)(\epsilon_r, \mu_r)_0, (\epsilon, \mu)_\theta = (1 - a/r)^{-1}(\epsilon_\theta, \mu_\theta)_0, (\epsilon, \mu)_z = (1 - a/b)^{-2}(1 - a/r)(\epsilon_z, \mu_z)_0$ are curved in exactly the same way to produce a cloak around a cylindrical object of radius a , centred at the origin [4].

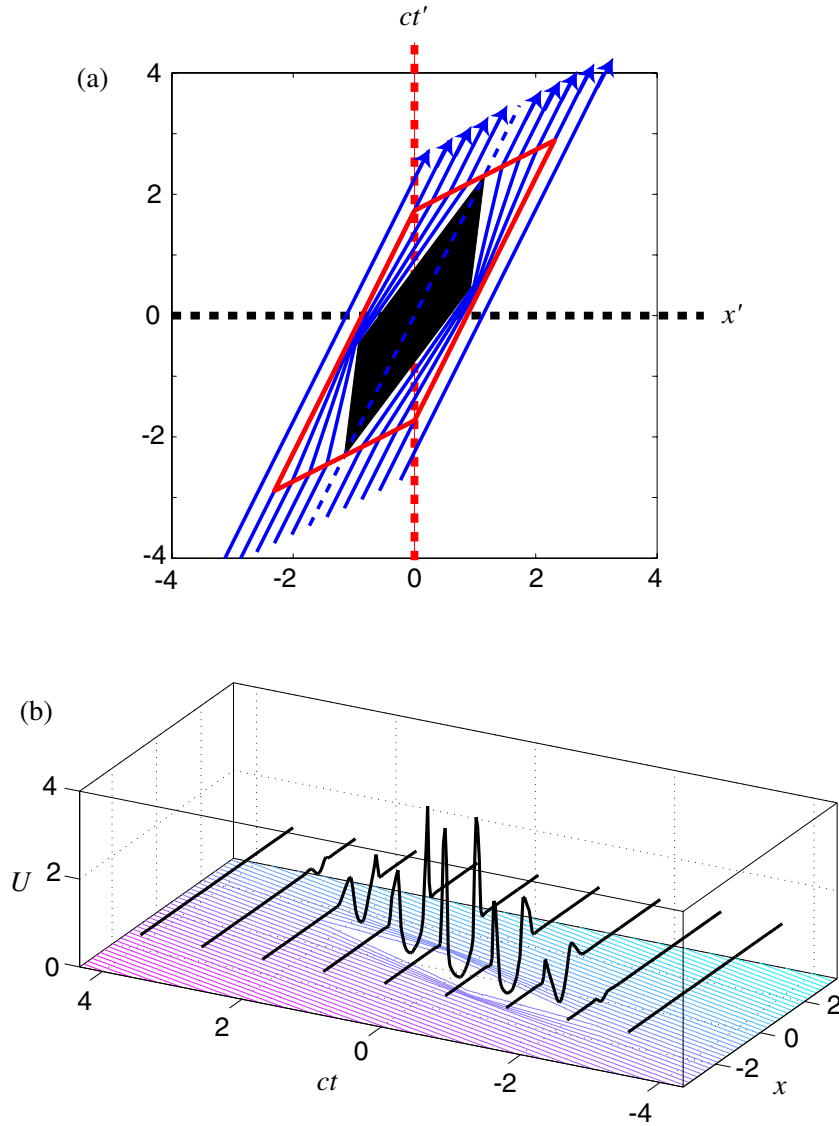


Figure 2. (a) Construction of a sub-luminal spacetime cloak. As with figure 1(b) the map $(x, t) \rightarrow (x', t')$ creates a void near the spacetime origin. However, the base space is filled with a medium of refractive index n rather than vacuum, so that prior to the transformation light rays are straight lines of gradient n . The transformation is a composition of a Lorentz boost, $(x, t) \rightarrow (\bar{x}, \bar{t})$, with velocity $v = c/n$, followed by applying a ‘curtain map’ $\bar{x} = [(\frac{\delta+|ct|}{\delta+n\sigma})(\bar{x} - \text{sgn}(\bar{x})\sigma) + \text{sgn}(\bar{x})\sigma]$, $\bar{t}' = \bar{t}$, followed by an inverse Lorentz transformation, $(\bar{x}', \bar{t}') \rightarrow (x', t')$. (See the appendix for further details.) Shown is the coordinate transformation $(x, t) \rightarrow (x', t')$ with $\sigma = 1$, $n = 2$, $\delta = 0.5$ for the curtain map. The deformed rays all have positive gradients (i.e. propagate forwards in time) and have speed $\leq c$. The coordinate transformation $(x, t) \rightarrow (x', t')$ defines the spacetime transformation that yields equations (1) and (2) with $t = t'$. Alternatively, a material with the required $\epsilon(x, t)$, $\mu(x, t)$, $\beta(x, t)$, enables the spacetime cloak to be realized in an actual medium. (b) Electromagnetic energy density for various times for the map of (a). The hard edges of the curtain map defined above have been softened by the use of mollifier functions (see the appendix), thus avoiding unrealistic instantaneous switching. The intensity is null at events near the spacetime origin.

For our spacetime cloak we consider spacetime transformations $(x, t) \rightarrow (x', t')$ and restrict to light propagating forwards along the x axis. In figure 1(b), for example, a spacetime transformation is carried out in the (x, t) plane that is analogous to the spatial transformation carried out in the (x, y) plane of figure 1(a). In contrast to the spatial cloak, where the direction of propagation in the x - y plane is arbitrary, the vacuum light rays must follow the straight lines $x = ct + \text{const}$. These rays are then mapped under the transformation to the curved rays shown so that the events within the disc surrounding the origin are avoided by the new rays. The new

light trajectories are actualized when they propagate through a suitable inhomogeneous time-dependent medium, and will then curve around the event occurring at $x = 0, t = 0$. As discussed in the appendix, the form of Maxwell’s equations (equations (1) and (2), but now with the addition that $t \rightarrow t'$) is again preserved provided the equivalent medium is magneto-electric with the transverse field components obeying

$$\begin{pmatrix} D_y \\ D_z \end{pmatrix} = \epsilon(x, t) \begin{pmatrix} E_y \\ E_z \end{pmatrix} + \beta(x, t) \begin{pmatrix} H_z \\ -H_y \end{pmatrix} \quad (3)$$

$$\begin{pmatrix} B_y \\ B_z \end{pmatrix} = \beta(x, t) \begin{pmatrix} -E_z \\ E_y \end{pmatrix} + \mu(x, t) \begin{pmatrix} H_y \\ H_z \end{pmatrix}. \quad (4)$$

The medium is thus in general required to be magneto-electric. In contrast to the spatial cloak, where light curves in the (x, y) plane, ‘curving’ light in the (x, t) plane now refers to different parts of the light paths speeding up and slowing down. The light proceeds only along the x axis and does not curve in space. As a consequence, events in the neighbourhood of the event at $(x, t) = (0, 0)$ are never suspected by an observer located sufficiently far to the right. In fact, all events in the y - z plane that occur near $(x, t) = (0, 0)$ are similarly undetected. The concealment results from various parts of the light distribution slowing down and speeding up so as to avoid events near the spacetime origin. The intensity along the x axis for different times is shown on the right of figure 1(b), indicating the formation of the intensity null that is fully developed at $t = 0$. For later times the null closes up, restoring a uniform intensity distribution so that the observer at $x = a$ records a constant intensity for all time. Since the events near $(x = 0, t = 0)$ are never illuminated, the observer to the right is unaware of their existence. In fact, all events in the (y, z) plane that occur near $(x = 0, t = 0)$ are concealed, so that effectively a spacetime corridor is opened along which non-radiating events, such as the movement of matter, exchange of information, etc, can occur undetected.

The spacetime cloak of figure 1(b) is symmetric, in that uniform illumination from the right will avoid the same cloaked events for an observer located sufficiently far to the left. As conceived by the symmetric transformation in the x - t plane of figure 1(b), some rays in the medium will be required to have a phase speed exceeding the vacuum speed of light (e.g. ray A in the inset) and some rays very close to the cloaked region (e.g. ray B) propagate backwards in time. Superluminal ray trajectories, which occur in purely spatial cloaks and in plasma propagation above the plasma frequency, are characterized by the phase velocity exceeding the vacuum speed of light, and are known to be compatible with special relativity [5]. Waves travelling backwards in time are precisely the interpretation of waves undergoing negative phase velocity propagation, or negative refraction, where the reversal of time in the wave’s phase $\phi = \mathbf{k} \cdot \mathbf{r} - \omega t$ is equivalent to the reversal of the direction of the wavevector \mathbf{k} with respect to electromagnetic flux [6]. However, both superluminal and negative phase waves, though accessible through appropriate metamaterials design, can, through similar design ingenuity, both be avoided in the spacetime cloak, as we discuss next.

3. Simplified designs

Referring to figure 2, the coordinate transformation is now carried out against a background of a uniform medium of refractive index $n > 1$, rather than vacuum. The complete transformation consists of a Lorentz transformation into a frame in which the rays are vertical, opening the void via a ‘curtain map’, followed finally by an inverse Lorentz transformation back into the medium rest frame (see the appendix for further details). The construction is such that

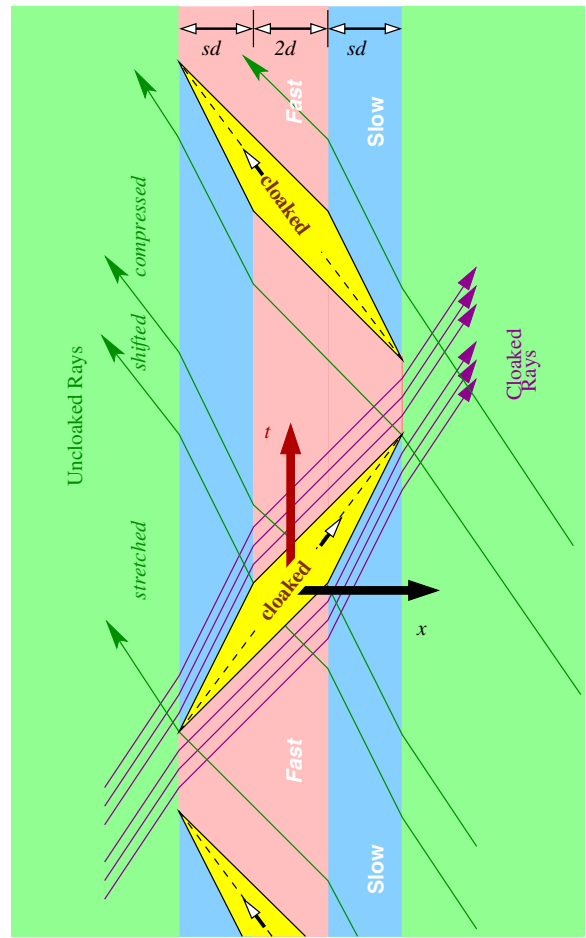


Figure 3. Spacetime cloak based on refractive index switching. (a) The base space is a medium of refractive index n with an inner region of width $2d$ surrounded by two outer regions of extent sd identified. For $-(1+s)d \leq x < 0$ the transformation is given by $x' = (s+2)(s+1)^{-1}x + d$ for $t \leq nx$ and by $x' = s(s+1)^{-1}x - d$ for $t > nx$. For $0 \leq x < (s+1)d$ the transformation is given by $x' = s(s+1)^{-1}x + d$ for $t \leq nx$ and by $x' = (s+2)(s+1)^{-1}x - d$ for $t > nx$. All other regions are unaffected. All rays are sub-luminal provided $n(1+s)(s+2)^{-1} < 1$. Unlike the cloak of figure 2(a), a single cloaking operation does not return the system to its original state. Alternating cloak operations as shown permits periodic restoration of the medium to its original state by interleaving a cloak operating in the reverse direction.

all rays in figure 2(a) travel slower than the vacuum speed of light. The plots of figure 2(b) show a detailed calculation of the electromagnetic energy density in the equivalent medium defined by the transformation, where the hard edges of the map have been mollified so that the medium is not switched instantaneously. The energy null which develops as the cloak is switched on becomes zero at events near the spacetime origin. Unlike the cloak of figure 1(b), this spacetime cloak only works for light travelling in the $+x$ direction. An observer to the left of the origin does see the cloaked events, though time separations between the cloaked events are first speeded up, then time-shifted, and then finally slowed down, before progression returns to normal.

The magneto-electric parameter, $\beta(x, t)$, in equations (3) and (4) arises almost invariably whenever the transformation

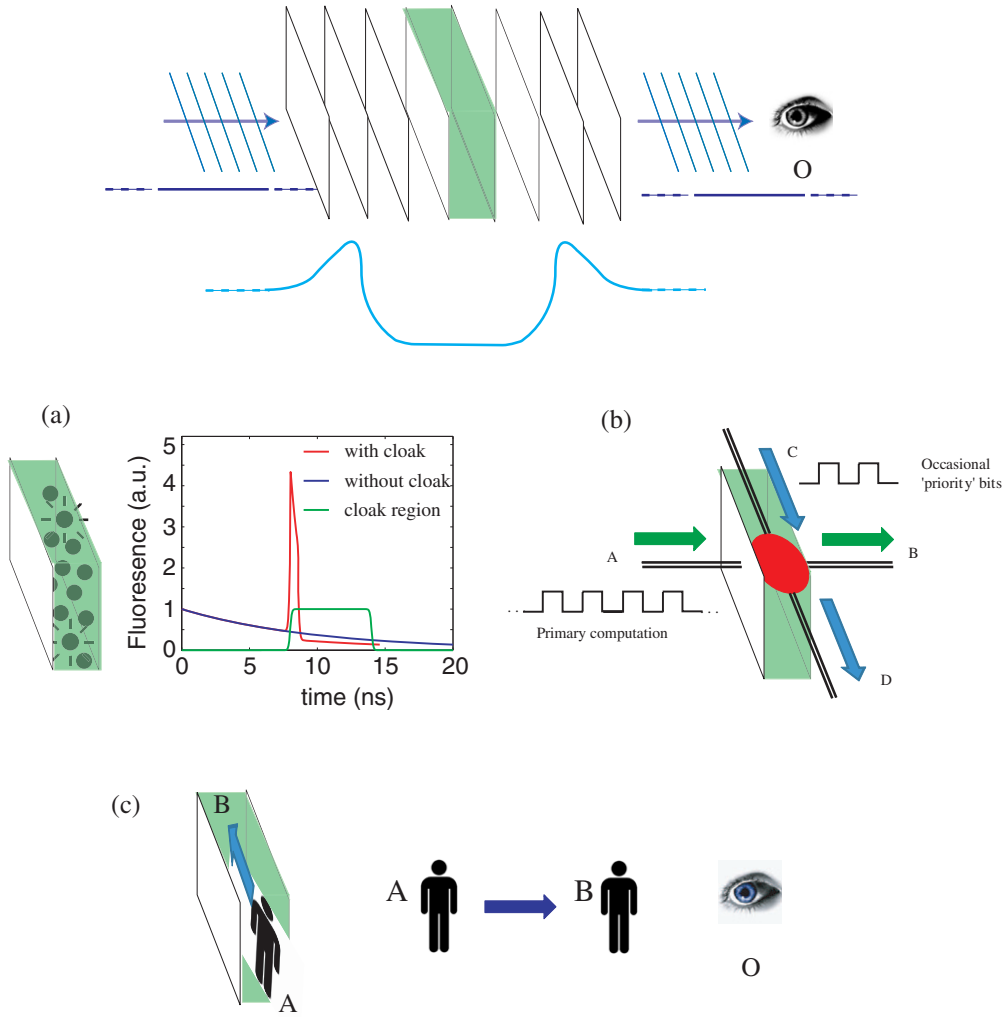


Figure 4. Realization of the spacetime cloak as a set of addressable planes with metallic inclusions. Applications based on how the central region is filled as the intensity null passes over: (a) excited atoms. The exponential fluorescence decay is observed with the emissions during cloak operation all apparently occurring simultaneously, resulting in an intensity spike, followed by resumption of the exponential decay. (b) ‘Interrupt-without-interrupt’. An occasional signal (channel AB) and a primary computation (channel CD) converge at the same physical node. Suppose it is required to process the occasional signal, without interrupting the primary computation. The cloak operation opens a temporal window via which the occasional signal can be processed, whilst the primary computation is reconstituted after the cloak operation to appear as if uninterrupted. (c) A Star Trek transporter. Movement of non-radiating matter whilst the cloak is operating appears to the observer O to have moved from A to B instantaneously.

$(x, t) \rightarrow (x', t')$ mixes space and time. This occurs, for example, in a medium of refractive index n moving with velocity v wherein the constitutive relations revert to the Minkowski form [7], the magneto-electric parameter then being given by

$$\beta(x, t) = \frac{v(x, t)}{c} \left[\frac{n^2 - 1}{1 - n^2 v^2(x, t)/c^2} \right]. \quad (5)$$

The spacetime cloak therefore involves the construction of a metamaterial that mimics propagation in a medium with a velocity $v(x, t)$. The occurrence of the null can then be understood via the velocity addition formula $v'(x, t) = [v(x, t) + c/n]/[1 + v(x, t)/(cn)]$, where $v'(x, t)$ is the velocity of light in the moving medium. Near $t = 0$, the effective medium is arranged so that $v(x, t)$ is negative for $x < 0$

and positive for $x > 0$. The trailing and leading parts of the light are thus respectively slowed down and speeded up to produce the required intermediate null. Non-reciprocal bi-anisotropic metamaterials that mimic the constitutive relations for a moving medium have been studied by Tretyakov [8], who has shown that such media can in principle be constructed from small magnetized ferrite spheres combined with planar-chiral metallic inclusions [9].

A spacetime cloak with a transformation of the form $t' = t, x' = x'(x, t)$ is illustrated in figure 3 which only requires manipulation of $\epsilon(x, t)$ and $\mu(x, t)$. The transformation is devised such that the spatio-temporal variation of these parameters matches, so that the impedance is uniform. The refractive index, however, is switched from high to low before the central region, and from low to high after the central region. The price of this simplified design is that the index

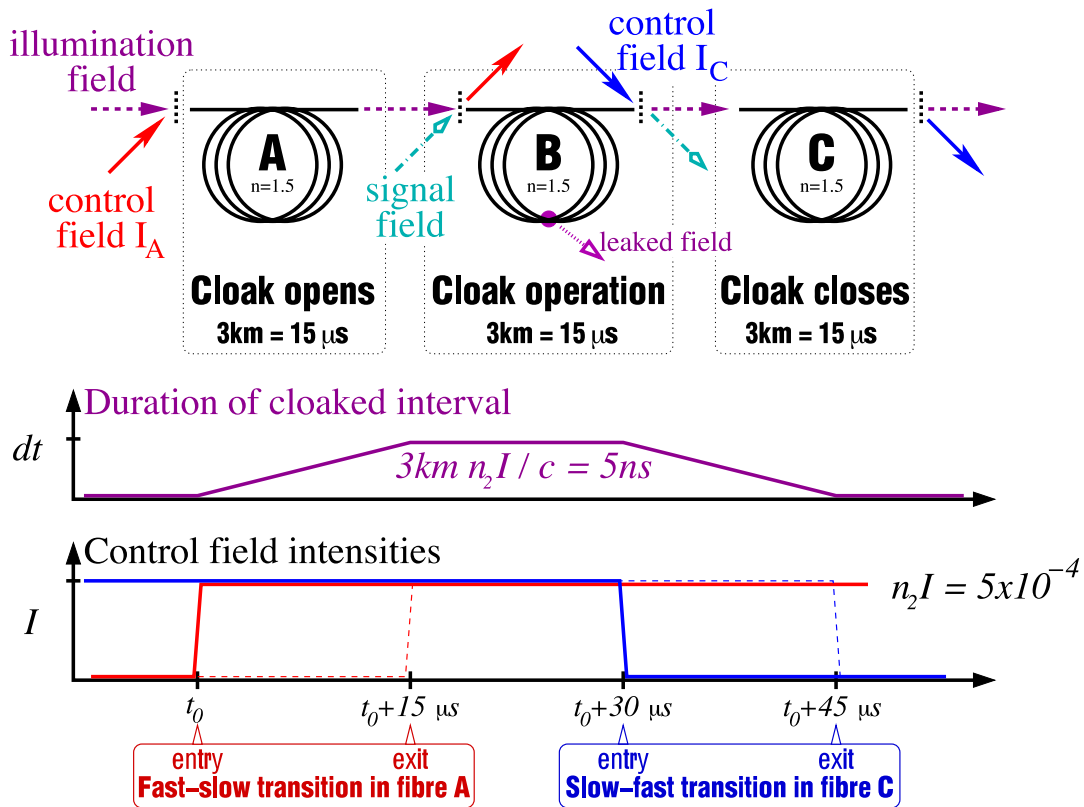


Figure 5. Spacetime cloak based on exploiting the nonlinear refractive index properties of optical fibres. The cloak is opened in fibre A, is held open in fibre B (which contains the core cloak region) and is closed again in fibre C. Two control fields I_A and I_C modulate the refractive index in A (C) from low to high (high to low). The much weaker illumination field experiences the index changes required for the cloak illustrated in figure 3. A signal field injected directly into fibre B is present before, during and after the cloaking from the illumination field. By monitoring the signal as it is coupled out, the spacetime modulation on it from the illumination and the effect of nonlinear refractive index can be measured. Changes to the signal field during the cloaking period leave no signature on the illumination field. Timing and synchronization is only critical in ensuring that the top right apex of the cloaking diamond (figure 5) aligns with the bottom-left corner at an angle given by the average phase speed of the signals. Fibre parameters: silica refractive index ~ 1.5 ; control field intensity set to yield an index change $\Delta n = n_2 I \sim 5 \times 10^{-4}$.

changes persist in the switched regions, so that, unlike the previous designs, the medium surrounding the spatial origin is not returned to its original state after the cloak has operated. Periodic switching, as shown in figure 3, returns the medium to its original state every other cycle and, moreover, permits bi-directionality by alternating the forward cloak with an identical one for light travelling in the $-x$ direction. If the index is switched without any impedance matching (as was done for the first demonstrations of the spatial cloak [4]), then reflections will occur at the interfaces. The observer to the right would consequently notice abrupt changes of the overall field intensity, but still remain ignorant of the events occurring within the cloaked period.

4. Demonstrations

What effects might be observable using the spacetime cloak? Figure 4 shows a collage of possibilities based around what happens in the $y-z$ plane in the region near $x = 0$ where/when the cloak operates. Filling this region with excited atoms (figure 4(a)) will cause the standard exponential fluorescence decay to be modified by the cloak, as emissions within

the cloaked period will all appear to happen simultaneously to the distant observer to the right. The observed spike could be a useful experimental signature, since the required cloaking period need only be comparable to the spontaneous lifetime ($\sim ns$), corresponding to a dimension of ~ 30 cm for the cloaked central region. Measurements of the coherence and statistics of light signals emitted by such a cloaked source will also yield anomalous results, even those of a quantum origin such as photon bunching or anti-bunching. Figure 4(b) proposes a specific application in signal processing where two channels, proceeding along the x and y directions respectively, cross at $x = 0$. Through the operation of the spacetime cloak occasional signals along the y channel can be processed as a priority, whilst processing and computation proceeds seamlessly, without interruption, along the x channel, achieving effectively an ‘interrupt-without-interrupt’. Finally, the spacetime cloak can achieve the illusion of a matter transporter (figure 4(c)), in which an object moves from (y_1, z_1) to (y_2, z_2) during the cloak’s operation. The observer to the right sees the object disappear at (y_1, z_1) to then instantaneously reappear at (y_2, z_2) .

It is emphasized that all the above demonstrations only require light propagation in one dimension since producing a temporary null in a collimated wave is all that is necessary to conceal events near $x = 0$ in the y - z plane for a limited period. However, it is clear that the spacetime cloak concept can be extended to more spatial dimensions where the illuminating light is not collimated. A spherical wave from a point source, for example, illuminates a set of events lying on a sphere at radius R from the origin. A spacetime cloak concealing these events will do so for distant observers positioned at any viewing angle.

5. Implementation

How practical is it to build a spacetime cloak? As with the original proposals for a spatial cloak, we have neglected dispersion, whose effect in this case will be to constrain the rate at which the cloak can be opened and closed. Figure 5 shows a scheme to implement the design of figure 3 using nonlinear optical fibres. The cloak is opened in fibre A, is held open in fibre B (which contains the core cloak region) and is closed again in fibre C. The primary operation is achieved by means of two control fields whose intensity allows the refractive index in A (C) to be modulated from low (high) to high (low). This is achieved using the nonlinear refractive index properties of the fibre (e.g. silica). The illumination field is affected by the refractive index shifts (i.e. by the cloak), but should have an intensity modulation far weaker than that used by the control fields. The cloak opens up a spacetime gap in the intensity profile of the illumination field. This gap could be detected by monitoring leaky spot(s) on fibre B, and looking for the intensity dip(s) in the illumination. Alternatively, a signal field could be injected directly into fibre B, and would therefore always be present, i.e. before, during and after the cloaking from the illumination field. By monitoring the signal as it is coupled out, the spacetime modulation on it from the illumination and the effect of nonlinear refractive index could be measured. For a brief period, the presence of the signal field will be undetectable to the surveilling illumination field and changes to the signal field during the cloaking period would leave no signature on the illumination field. The group velocity of the on/off transitions must be no slower than the slow phase velocity and no faster than the fast phase velocity. This ensures that the control/cloaking index transitions always remain inside the cloak. Good dispersion control would be required (probably using photonic crystal fibres): If the various fields all have distinct frequencies or polarizations then they can be coupled in and out efficiently and distinctly. The fibre parameters illustrated in figure 5 are optimistic but accessible (though harmonic generation and Raman and Brillouin scattering would also need to be considered). Similarly, all coupling is assumed to be perfect, although slight leakage of control field A into fibres B and C should be tolerable.

6. Conclusion

We have introduced a fundamentally new concept of electromagnetic cloaking that exploits the transformation

optics algorithm in one space and one time dimension. The result is a cloak that operates in a fundamentally distinct way to the purely spatial, or object, cloak, in that it conceals events by curving light rays in spacetime rather than in space. Although this sets new challenges for metamaterials design, we have shown how these challenges can be minimized through judicious manipulation of the refractive index of the meta-medium. We have proposed a proof-of-principle design based on nonlinear fibre optics that could demonstrate the signature of spacetime cloaking, and how the concept could find practical applications in signal processing.

We are sure that there are many other possibilities that are opened up by our introduction of the concept of the spacetime cloak. Whilst the camera is said to never lie, it does not always record the whole truth.

Acknowledgments

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Appendix

A.1. Calculating the material parameters for the spacetime cloak

Calculating the required spatial and temporal material properties of the spacetime cloak, and the resultant distribution of electromagnetic energy, is most easily carried out using a covariant approach. We take Cartesian-Lorentz coordinates indexed as $\alpha = 0-3$, i.e. $\{x^\alpha\} = (ct, x, y, z)$. The source-free Maxwell equations in a linear medium responding instantaneously to electromagnetic excitation may be expressed as

$$\frac{\partial}{\partial x^\beta} (\chi^{\alpha\beta\mu\nu} F_{\mu\nu}) = 0, \quad (6)$$

where $F_{\mu\nu}$ is the electromagnetic field tensor given by (in units where $c = 1$)

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}, \quad (7)$$

The fourth-rank object $\chi^{\alpha\beta\mu\nu}$ relates $F_{\mu\nu}$ to $G^{\alpha\beta}$ according to

$$G^{\alpha\beta} = \chi^{\alpha\beta\mu\nu} F_{\mu\nu}, \quad (8)$$

where

$$G^{\alpha\beta} = \begin{pmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & H_z & -H_y \\ -D_y & -H_z & 0 & H_x \\ -D_z & H_y & -H_x & 0 \end{pmatrix}, \quad (9)$$

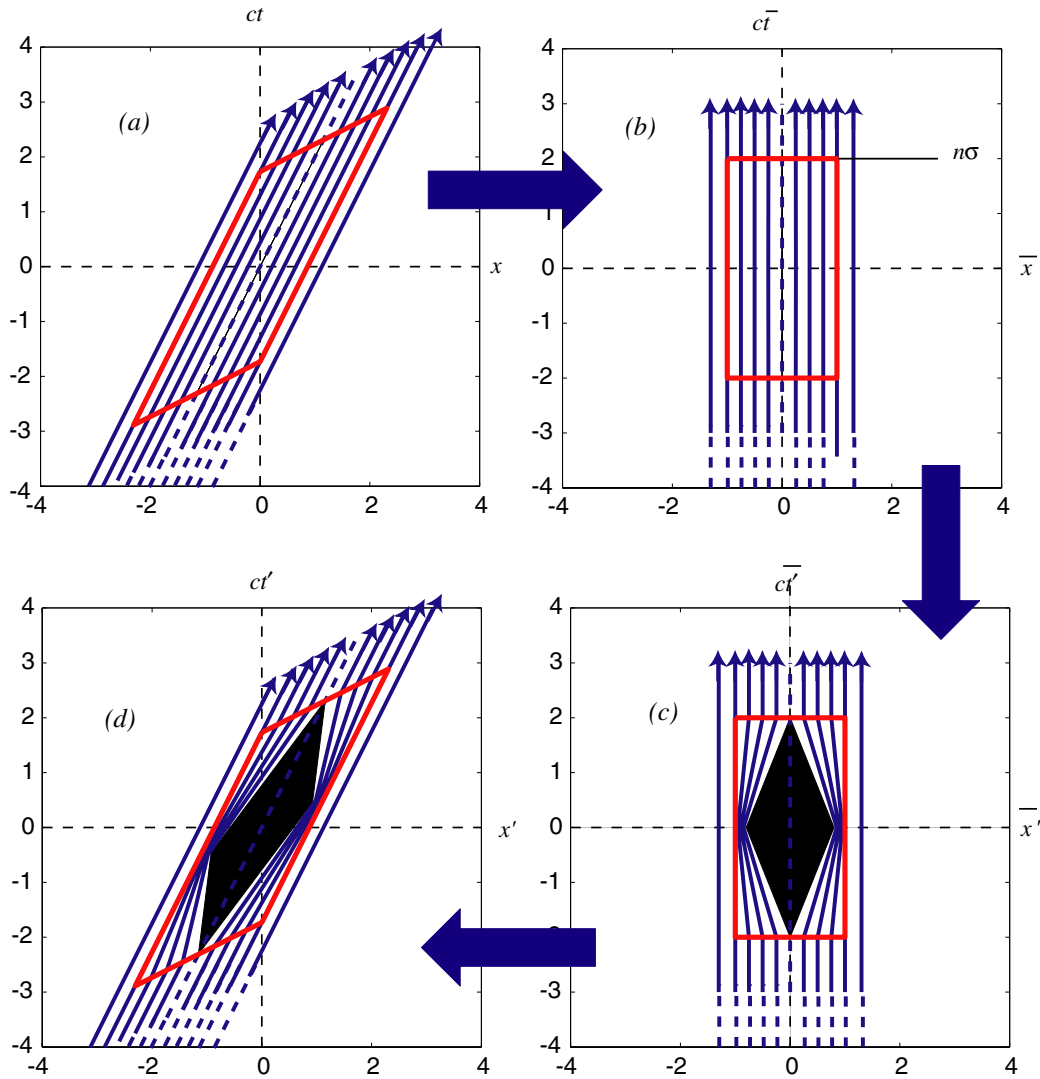


Figure 6. The detailed construction of the curtain map.

i.e. $G^{\alpha\beta}$ contains the fields \mathbf{D} and \mathbf{H} . The fourth-rank object $\chi^{\alpha\beta\mu\nu}$ contains the electromagnetic material parameters. In non-covariant notation we have

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\epsilon & \alpha \\ -\alpha^T & \mu^{-1} \end{pmatrix} \begin{pmatrix} -\mathbf{E} \\ \mathbf{B} \end{pmatrix}, \quad (10)$$

where T indicates transpose. In fact, $\chi^{\alpha\beta\mu\nu}$ is a fourth-rank tensor density and transforms according to [10]

$$\chi^{\alpha'\beta'\mu'\nu'} = |\det(L_\lambda^{\lambda'})|^{-1} L_\alpha^{\alpha'} L_\beta^{\beta'} L_\mu^{\mu'} L_\nu^{\nu'} \chi^{\alpha\beta\mu\nu}, \quad (11)$$

where $L_\alpha^{\alpha'} = \partial x^{\alpha'} / \partial x^\alpha$.

For a transformation restricted to the (x^0, x^1) plane and a base constitutive tensor for which the non-zero elements are $\chi^{0101} = \chi^{0202} = \chi^{0303} = -\epsilon$; $\chi^{2323} = \chi^{3131} = \chi^{1212} = \mu^{-1}$ we find the non-zero elements of $\chi^{\alpha'\beta'\mu'\nu'}$ to be

$$\begin{aligned} \chi^{0'1'0'1'} &= |L_\alpha^{\alpha'}|^{-1} (L_0^0 L_1^1 L_0^0 L_1^1 - L_0^0 L_1^1 L_0^0 L_1^1) \\ &+ L_1^0 L_0^1 L_0^0 L_1^1 - L_1^0 L_0^1 L_0^0 L_1^1 (-\epsilon) \equiv -\epsilon_{||} \end{aligned} \quad (12)$$

$$\begin{aligned} \chi^{0'2'0'2'} &= |L_\alpha^{\alpha'}|^{-1} (-\epsilon L_0^0 L_0^0 + \mu^{-1} L_1^0 L_1^0) \\ &= \chi^{0'3'0'3'} \equiv -\epsilon_{\perp} \end{aligned} \quad (13)$$

$$\chi^{2'3'2'3'} = |L_\alpha^{\alpha'}|^{-1} \mu^{-1} \equiv \mu_{||}^{-1} \quad (14)$$

$$\begin{aligned} \chi^{3'1'3'1'} &= |L_\alpha^{\alpha'}|^{-1} (-\epsilon L_0^1 L_0^1 + \mu^{-1} L_1^1 L_1^1) \\ &= \chi^{1'2'1'2'} \equiv \mu_{\perp}^{-1} \end{aligned} \quad (15)$$

$$\begin{aligned} \chi^{0'2'1'2'} &= |L_\alpha^{\alpha'}|^{-1} (-\epsilon L_0^0 L_0^1 + \mu^{-1} L_1^0 L_1^1) \\ &= -\chi^{0'3'3'1'} = \chi^{1'2'0'2'} = -\chi^{3'1'0'3'} \equiv \alpha. \end{aligned} \quad (16)$$

The material relations (10) can thus be expressed alternatively as

$$D_x = \epsilon_{||} E_x, \quad B_x = \mu_{||} H_x, \quad (17)$$

$$\begin{pmatrix} D_y \\ D_z \end{pmatrix} = (\epsilon_{\perp} + \alpha^2 \mu_{\perp}) \begin{pmatrix} E_y \\ E_z \end{pmatrix} + \alpha \mu_{\perp} \begin{pmatrix} H_z \\ -H_y \end{pmatrix}, \quad (18)$$

$$\begin{pmatrix} B_y \\ B_z \end{pmatrix} = \alpha \mu_{\perp} \begin{pmatrix} -E_z \\ E_y \end{pmatrix} + \mu_{\perp} \begin{pmatrix} H_y \\ H_z \end{pmatrix}. \quad (19)$$

Equations (18) and (19) are seen to be of the same form as equations (3) and (4).

A.2. The curtain map in detail

The curtain map of figure 2 consists of successive spacetime coordinate transformations $(x, t) \rightarrow (\bar{x}, \bar{t}) \rightarrow (\bar{x}', \bar{t}') \rightarrow (x', t')$ as shown in figure 6. Light propagating in a uniform medium of refractive index n is represented by the straight sub-luminal world lines of figure 6(a). The transition from figures 6(a) to 6(b) is a Lorentz boost with speed $v = c/n$ applied according to

$$\bar{x} = (1 - n^{-2})^{-1/2}(x - ct/n), \quad (20)$$

$$\bar{t} = (1 - n^{-2})^{-1/2}(t - x/nc). \quad (21)$$

This produces the vertical photon world lines of figure 6(b) wherein the rectangular box ($|\bar{c}\bar{t}| < n\sigma$, $|\bar{x}| < \sigma$) is identified. Within this rectangle a void region is opened up via the operation of a ‘curtain map’ defined via

$$\bar{x}' = \left(\frac{\delta + |\bar{c}\bar{t}|}{\delta + n\sigma} \right) [\bar{x} - \text{sgn}(\bar{x})\sigma] + \text{sgn}(\bar{x})\sigma, \quad (22)$$

$$\bar{t}' = \bar{t}, \quad (23)$$

with points outside the rectangle being mapped to themselves. The curtain map is shown for which $\sigma = 1$, $n = 2$, $\delta = 0.5$. Inverse Lorentz-transforming back to the medium rest frame, where the coordinates are now (x', t') , yields the obliquely opened curtain as shown in figure 6(d). The deformed photon world lines in figure 6(d) all have positive gradients (i.e. propagate forwards in time) and have speeds (determined by the gradient of the world line) less than the vacuum speed of light. The hard edges of the curtain map induced by the use of $|\cdot|$ and $\text{sgn}(\cdot)$ functions in equation (22) may be softened by the use of suitable mollifier functions as explained in the next section.

A.3. Mollifying the hard edges of the curtain map

As defined above, the composite map $(x, t) \rightarrow (\bar{x}, \bar{t}) \rightarrow (\bar{x}', \bar{t}') \rightarrow (x', t')$ is ‘hard-edged’, meaning that sections of the intensity at a given t' will change abruptly from zero inside the void region to a finite value outside. This would require unrealistic instantaneous changes in the material parameters at the edge of the cloak. In order to soften these edges we applied the following procedure.

In the vertical ray frame (i.e. figure 6(b)) the functions $\text{sgn}(\xi)$ and $|\xi|$ in equation (22) are replaced by $\tanh(\xi/\Delta)$ and $\xi \tanh(\xi/\Delta)$, respectively, where Δ is a small parameter. Note that, as $\Delta \rightarrow 0$, $\tanh(\xi/\Delta) \rightarrow \text{sgn}(\xi)$ and $\xi \tanh(\xi/\Delta) \rightarrow |\xi|$. For points inside the rectangle equations (22) and (23) are therefore replaced with

$$\bar{x}'_{\text{in}} = \left[\left(\frac{\delta + \bar{c}\bar{t} \tanh(\bar{c}\bar{t}/\Delta)}{\delta + n\sigma} \right) [\bar{x} - \tanh(\bar{x}/\Delta)\sigma] + \tanh(\bar{x}/\Delta)\sigma \right], \quad (24)$$

$$\bar{t}'_{\text{in}} = \bar{t}. \quad (25)$$

The mollified curtain map of equations (24) and (25) is continuous and differentiable at the origin. Points outside the rectangle ($|\bar{x}| > \sigma$, $|\bar{c}\bar{t}| > n\sigma$) are left unchanged as before (i.e. $\bar{x}'_{\text{out}} = \bar{x}$, $\bar{t}'_{\text{out}} = \bar{t}$). To avoid discontinuities at the boundary of the box, the maps inside the rectangle and outside the rectangle are graded into each other by means of the mollified rectangular ‘top-hat’ function:

$$\rho(\bar{x}, \bar{c}\bar{t}) = \frac{1}{4} \left[\tanh\left(\frac{\bar{c}\bar{t} + n\sigma}{\Delta}\right) - \tanh\left(\frac{\bar{c}\bar{t} - n\sigma}{\Delta}\right) \right] \times \left[\tanh\left(\frac{\bar{x} + \sigma}{\Delta}\right) - \tanh\left(\frac{\bar{x} - \sigma}{\Delta}\right) \right]. \quad (26)$$

Note that, as $\Delta \rightarrow 0$, equation (26) becomes a rectangular top-hat function. The maps inside and outside the rectangle are then graded into each other according to

$$\bar{x}' = \rho \bar{x}'_{\text{in}} + (1 - \rho) \bar{x}'_{\text{out}}, \quad (27)$$

$$\bar{t}' = \bar{t}. \quad (28)$$

This procedure ensures that the resultant map $(\bar{x}, \bar{t}) \rightarrow (\bar{x}', \bar{t}')$ defined by combining equations (27) and (28) with equations (24)–(26) is continuous at the boundary of the rectangle. The complete map $(x, t) \rightarrow (x', t')$ is then specified via the Lorentz transformation of equations (20) and (21) sending $(x, t) \rightarrow (\bar{x}, \bar{t})$, the mollified curtain map of equations (27) and (28) sending $(\bar{x}, \bar{t}) \rightarrow (\bar{x}', \bar{t}')$, followed by the inverse Lorentz transformation sending $(\bar{x}', \bar{t}') \rightarrow (x', t')$. The composite map $(x, t) \rightarrow (x', t')$, with $\Delta = 0.08$, was used to generate figure 2 in this paper by calculating numerically the transformed material parameters using equations (12)–(16), and then calculating the electromagnetic energy density according to

$$U = \frac{1}{2}(\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}). \quad (29)$$

References

- [1] Pendry J B 2000 Negative refraction makes a perfect lens *Phys. Rev. Lett.* **85** 3966
- [2] Pendry J B, Schurig D and Smith D R 2006 Controlling electromagnetic fields *Science* **312** 1780–2
- [3] Leonhardt U and Philbin T G 2006 General relativity in electrical engineering *New J. Phys.* **8** 247
- [4] Schurig D, Mock J J, Justice B J, Cummer S A, Pendry J B, Starr A F and Smith D R 2006 Metamaterial electromagnetic cloak at microwave frequencies *Science* **314** 977–80
- [5] Brillouin L 1960 *Wave Propagation and Group Velocity* 1st edn (New York: Academic)
- [6] Pendry J B 2009 Time reversal and negative refraction *Science* **322** 71–3
- [7] McCall M W 2007 Relativity and mathematical tools: waves in moving media *Am. J. Phys.* **75** 1134–40
- [8] Tretyakov S A, Nefedov I S and Alitalo P 2008 Generalized field-transforming metamaterials *New J. Phys.* **10** 115028
- [9] Tretyakov S A 2009 On a possibility to imitate media moving with superluminal velocity *3rd Int. Conf. on Advanced Electromagnetic Materials in Microwaves and Optics* London
- [10] Post E J 1997 *Formal Structure of Electromagnetics* (New York: Dover)